

A note on the notion “statistical symmetry”

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Abstract

A critical review is presented on the most recent attempt to generally explain the notion of “statistical symmetry”. This particular explanation, however, is incomplete and misses one important and essential aspect. The aim of this short note is to provide this missing information and to clarify this notion on the basis of a few instructive examples.

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1. Introduction

An attempt to explain the notion of “statistical symmetry” was recently made in Waclawczyk & Oberlack (2015). Although stated correctly that such a name has “been used in other areas of science, e.g., in the study of dynamical systems [6]=Aubry & Lima (1995), statistical physics [7]=Birman & Trebin (1985), or even sensory coding [8]=Turiel & Parga (2003)” [p. 1], and that “even if one particular field (or image) does not verify the symmetry, it can be observed over a large ensemble of fields (or images)” [p. 1], these statements still miss one important fact, namely to mention that if the statistical description is based on an underlying deterministic theory, then every statistical symmetry must have some kind of cause from which it can emerge, whereby the cause itself certainly need not to be a symmetry transformation. The obvious reason is that since the deterministic equations due to their spatially nonlocal and temporally chaotic behavior induce the statistical equations, and not vice versa, and since any symmetry in physics is always defined as a transformational process, namely as an invariant transition from one state to another, this occurrent transition, i.e. the transition itself must then have some deterministic cause if a symmetry is to be observed statistically.

Hence, the cause for a statistical symmetry must be encoded as a change on the lower deterministic (fluctuating) level such that for a large ensemble of fields (or images) a symmetry can be observed. Important to note here again is that the cause itself need not to be symmetry, but at least there must be an occurring change on the fluctuating level, namely from one state to another such that on the averaged level this change can then be observed as a symmetry. In other words, a statistical symmetry is based on an active construction process resulting from a (mostly non-invariant) change of its underlying deterministic system. All statistical symmetries derived and discussed in the above cited studies (Aubry & Lima, 1995; Birman & Trebin, 1985; Turiel & Parga, 2003) are of this type, in that they all have a cause which all originate from a non-invariant change in the underlying deterministic system — in particular, because they are all based on pure space-time coordinate transformations, which not only transform the deterministic but also the induced statistical state, however, such that on the mean level a symmetry can be observed.

To be explicit, the study by Aubry & Lima (1995) investigates the connection, i.e. the cause and effect between the (microscopic) spatiotemporal symmetries of a space-time function and the (macroscopic) symmetries of a statistical quantity, as e.g. for the two-point

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correlation, while Birman & Trebin (1985) defines “statistical symmetries” as “symmetries in the usual sense” [p. 388], namely as deterministic coordinate transformations (translations, rotations, reflections, etc.) which only get the label ‘symmetry’ as soon as they leave the configurational or temporal average invariant. The third cited study by Turiel & Parga (2003) considers e.g. a deterministic spatial scaling transformation (via a wavelet transformation) in every single image which then induces a local invariance of a fixed scale in the ensemble of all images, i.e. the cause of this statistical symmetry is the (non-invariant) transformation of each single image which then induces the effect of a symmetry in the ensemble of all images.

The same is also true for the statistical symmetry discussed in Waławczyk & Oberlack (2015), which actually has been taken from Aubry & Lima (1995) [p. 794], however, just reformulated for an unsteady Poiseuille flow in a channel. Indeed, the discussed statistical reflection symmetry for the mean velocity profile about the center plane $y = 0$ in Waławczyk & Oberlack (2015) [p. 1] has a cause, namely the spatial reflection transformation itself, which changes the instantaneous state $U(x, y, z)$ into a different state $U(x, -y, z) \neq U(x, y, z)$ with equal probability, such that within an ensemble of the instantaneous velocity field this change or transformation emerges then as a symmetry on the mean level: $\langle U \rangle(-y) = \langle U \rangle(y)$.

Another simple example is to consider a fully developed turbulent channel flow which is statistically stationary in time. Although the time shift transformation is not a symmetry of the instantaneous velocity field, it nevertheless serves as the cause to emerge as a symmetry in the correspondingly induced statistical field. To be explicit, let us view this example from the perspective of a numerical simulation: Imagine we have a DNS result of a fully developed turbulent channel flow. Within this flow regime we then collect an ensemble of the instantaneous velocity field, say, of the streamwise component at different times, however, with a time-interval always sufficiently large in order to ensure statistical independence. If one now takes the ensemble average over these different instantaneous field realizations, we observe, as an effect, that the flow statistics is independent of time. But the cause of this effect is the preceding process of evaluating or choosing the instantaneous velocity field at different times, which itself just represents a deterministic time shift transformation, in that the considered DNS field successively gets transformed to different times in order to obtain the elements of this particular ensemble. In other words, the statistical time translation symmetry emerges as an effect from the cause of transforming the underlying deterministic (instantaneous) field to different times.

A further instructive example is discussed in the next section, where we even present a more detailed and at the end also a more general investigation of this cause-and-effect issue for statistical symmetries.

Currently, within incompressible fluid mechanics, no other or new statistical symmetries for the Euler and Navier-Stokes equations are known yet than the classical ones listed through Eqs. (8)-(13) in Waławczyk *et al.* (2014), which all have their origin in the deterministic equations. In clear contrast, of course, to the *newly* proposed statistical symmetries given in Waławczyk *et al.* (2014) by Eq. (14) and Eq. (16) for the multi-point correlation functions, or by Eq. (42) and Eq. (63) for the PDFs, which are emerging without any cause at all. As shown in Frewer *et al.* (2014a,b), one even runs into a contradiction as soon as one tries to establish a cause of any kind. The problem is that *all* multi-point correlation functions and *all* multi-point PDFs get scaled or translated by a spatially constant quantity for which at the same time the coordinates and all deterministic fields stay invariant.[†] Hence, since no cause

[†]The magnitude of this problem also becomes apparent through the following contradictory statement made in Waławczyk & Oberlack (2015): Saying on p. 2 after Eq. (5) that “*Here, the single [deterministic] velocity fields v and U are not rescaled [or changed] but the share of laminar solutions in the ensemble changes. This could be caused, e.g., by the change of initial conditions or external disturbances*” is obviously fictitious and unphysical. Because, how is it possible that if initial conditions change or external disturbances appear that then the underlying deterministic fields do not change? This conflict in assumption ultimately also explains their inconsistent result summarized on p. 4: “*After such [symmetry] transformations the statistics change, although the [underlying] instantaneous velocities U in separate realizations of the flow are not transformed (i.e., they are not rescaled or translated)*”, constitutes a clear inconsistency, because again, from where should this change on the statistical level come from when the underlying deterministic system itself is not changing?

for those new statistical symmetries can be constructed, they obviously violate the classical principle of causality as we fully elaborated in our Comment (Frewer *et al.*, 2015, 2014a). This unphysical behavior can also be clearly seen when comparing to DNS data (Simens *et al.*, 2009; Borrell *et al.*, 2013); a detailed summary of this analysis is given e.g. in Frewer (2015). The only conclusion to be drawn from this negative result, is to discard these newly proposed statistical symmetries in order to avoid any misleading conclusions in the theory of turbulence.

2. The causality in a statistical symmetry at the example of a random walk

Without loss of generality, we consider the spatial 1D case for simplicity. As we know, the problem ‘random-walk’ can be discussed mathematically in two *equivalent* ways, either directly as a dynamical process through a stochastic variable, or indirectly as an evolutionary process through a probability distribution characterizing the dynamics of this stochastic variable.[†] In both approaches one has to distinguish between the discrete (fine-grained) random walk on a space-time lattice and the continuous (coarse-grained) random walk in a space-time continuum in which the emergent concept of diffusion arises.

2.1. The discrete (fine-grained) random walk

The random walk of a spatially 1D moving particle on a space-time lattice can be investigated

- (i) either as the recursive stochastic process

$$m_{N+1} = m_N + \xi_N, \quad (2.1)$$

where m_N is the stochastic variable for finding the particle at position m after N successive steps from the initial position, say $m = 0$, being forced by a random variable ξ_N , which in each step N can only take one of two possible values $\xi_N = \pm 1$, where for simplicity we assume that both outcomes occur with equal probability $1/2$,

- (ii) or, equivalently, by the recursive probability (master) equation

$$P_{N+1}(m) = \frac{1}{2} \cdot P_N(m-1) + \frac{1}{2} \cdot P_N(m+1), \quad (2.2)$$

to find the distribution $P_N(m)$ that after N steps the particle can be found at position m .

In both cases (i) and (ii) the variables $N \geq 0$ and m are integers, such that $-N \leq m \leq N$ is always satisfied. The solution of (2.1) is given by

$$m_N = \sum_{i=0}^{N-1} \xi_i, \quad (2.3)$$

with a vanishing expectation value or first moment $\langle m_N \rangle = 0$ and a variance or second moment which equals to the number of steps performed $\langle m_N^2 \rangle = N$, due to the zero mean and due to the statistical independence of the random variable ξ_i respectively: $\langle \xi_i \rangle = 0$ and $\langle \xi_i \cdot \xi_j \rangle = \delta_{ij}$. The solution to equation (2.2) is given by the binomial distribution

$$P_N(m) = \frac{N!}{\left(\frac{N+m}{2}\right)! \left(\frac{N-m}{2}\right)!} \cdot \left(\frac{1}{2}\right)^N, \quad (2.4)$$

[†]The theory on which this section is based can be found in any standard text book on stochastic processes, see e.g. Gardiner (1985); Salinas (2001); Mahne *et al.* (2009).

which only gives a physical solution to the random walk if $N_{\pm} = \frac{N \pm m}{2} \in \mathbb{N}_0$, where N_+ is the number of steps to the right and $N_- = N - N_+$ the number of steps to the left in order to end after N steps at position m . The solution (2.4) is already properly normalized

$$\sum_{N_+=0}^N \frac{N!}{N_+!(N-N_+)!} \left(\frac{1}{2}\right)^N = 1, \quad (2.5)$$

and gives the same values for the moments as solution (2.3): $\langle m \rangle = 2\langle N_+ \rangle - N = 0$ and $\langle m^2 \rangle = \langle (2N_+ - N)^2 \rangle = N$. Finally, let us introduce, instead of the non-dimensional integer variables N and m , the real physical variables t and x respectively:

$$t = N \cdot \tau, \quad x = m \cdot l, \quad (2.6)$$

where τ is a constant time interval between *all* successive steps of the same length l . That means, $P_N(m)$ (2.4) can then be interpreted as the probability of finding the particle at the physical position $x = m \cdot l$ at the time $t = N \cdot \tau$. In the following, the parameters l and τ are arbitrary but fixed (finite) values. Now, let's consider the following point transformation of variables

$$\tilde{t} = c^2 \cdot t, \quad \tilde{x} = c \cdot x, \quad \tilde{P} = P, \quad \text{where } c \in \mathbb{Z}/\{0\}, \quad (2.7)$$

which induces the transformation in the original variables as

$$\tilde{N} = c^2 \cdot N, \quad \tilde{m} = c \cdot m, \quad \tilde{P} = P. \quad (2.8)$$

It is obvious that it's *not* a symmetry transformation of equations (2.1) and (2.2), since neither solution (2.3) nor solution (2.4) gets mapped to a new solution of equation (2.1) and equation (2.2) respectively, i.e. the mapped function of solution (2.3)

$$\tilde{m}_{\tilde{N}} \mapsto m_N = \frac{1}{c} \cdot \sum_{i=0}^{c^2 \cdot N - 1} \xi_i, \quad (2.9)$$

as well as the mapped function of solution (2.4)

$$\tilde{P}_{\tilde{N}}(\tilde{m}) \mapsto P_N(m) = \frac{(c^2 \cdot N)!}{\left(\frac{c^2 \cdot N + c \cdot m}{2}\right)! \left(\frac{c^2 \cdot N - c \cdot m}{2}\right)!} \cdot \left(\frac{1}{2}\right)^{c^2 \cdot N}, \quad (2.10)$$

are no longer solutions of equations (2.1) and (2.2) anymore.

2.2. The continuous (coarse-grained) random walk

The limit of large N and m , i.e. $N \gg \tau$ and $m \gg l$, is to be seen as a coarse graining process of the random walk. In general, coarse-graining is understood as a process which eliminates the “uninteresting” fast and small variables and keeps the coarse-grained variables with time and space scales much larger than the microscopic (fine-grained) scales τ and l . And exactly this is what this limit of large N and m does, for which then (i) the discrete (fine-grained) stochastic process (2.1) turns into the continuous (coarse-grained) 1D Wiener process

$$dx_t = \sqrt{2D} \cdot dw_t, \quad (2.11)$$

with its well-known statistical properties $\langle w_t \rangle = 0$ and $\langle w_t^2 \rangle = t$, and for which (ii) the discrete (fine-grained) probability distribution equation (2.2) turns into the associated continuous (coarse-grained) Fokker-Planck equation

$$\frac{\partial P}{\partial t} = D \cdot \frac{\partial^2 P}{\partial x^2}. \quad (2.12)$$

Both equations (2.11) and (2.12) describe the physical process of diffusion in a mathematical equivalent manner, where $D = l^2/2\tau$ denotes the diffusion coefficient.

Suddenly, in the limit of large $N \gg \tau$ and $m \gg l$, i.e. in the process of coarse-graining the defining system, the transformation (2.7), which keeps its validity as a variable transformation in this limit, turns into a *symmetry* transformation for the continuous (coarse-grained) equations (2.11) and (2.12). That means, *any* solution of (2.12) gets mapped into a new solution. For example, the properly normalized solution

$$P(t, x) = \frac{1}{\sqrt{4\pi Dt}} \cdot e^{-\frac{x^2}{4Dt}}, \quad (2.13)$$

which satisfies the initial condition $P(0, x) = \delta(x)$ as a Cauchy problem, gets mapped by transformation (2.7) to a new solution of (2.12)

$$\tilde{P}(\tilde{t}, \tilde{x}) \mapsto P(t, x) = \frac{1}{\sqrt{4\pi D \cdot c^2 \cdot t}} \cdot e^{-\frac{x^2}{4Dt}}, \quad (2.14)$$

satisfying a different, the transformed initial condition $\tilde{P}(0, \tilde{x}) \mapsto P(0, x) = \delta(x)/|c|$ with the normalizing constant $|c|$. And if, in addition, the symmetry coordinate transformation (2.7) would be augmented by the appropriate symmetry scaling in the dependent variable P as

$$\tilde{t} = c^2 \cdot t, \quad \tilde{x} = c \cdot x, \quad \tilde{P} = \frac{1}{|c|} \cdot P, \quad (2.15)$$

then the Cauchy initial-value solution (2.13) even turns into a *self-similar* solution under the combined scaling symmetry (2.15) admitted by equation (2.12). Also the general solution

$$x_t = x_0 + \sqrt{2D} \cdot w_t, \quad (2.16)$$

of its associated stochastic differential equation (2.11) forms a *self-similar* solution under the scaling symmetry (2.7), if the initial condition is placed at the origin: $x_0 = 0$. To explicitly see that the coordinate transformation (2.7) is admitted as a symmetry by equation (2.11), or that (2.16) is a self-similar solution under this scaling if $x_0 = 0$, it is necessary to realize that the defining time transformation $\tilde{t} = c^2 \cdot t$ induces the following scaling in the Wiener process (see e.g. Gaeta & Quintero (1999); Srihirun *et al.* (2007); Abdullin *et al.* (2014))

$$w_{\tilde{t}} = c \cdot w_t =: \tilde{w}_{\tilde{t}}, \quad (2.17)$$

that means, unlike the discrete (fine-grained) random walk, the continuous (coarse-grained) random walk is scale invariant, i.e. if w_t is a Wiener process then $c^{-1}w_{c^2t}$ is again a Wiener process.

2.3. Conclusion[†]

Although transformation (2.7) is a symmetry transformation only on the coarse-grained level (2.12) and not on the fine-grained level (2.2), the transformation (2.7) itself is nevertheless a valid variable transformation on both levels. And since the combined system (2.2) & (2.7) on the fine-grained level uniquely induces the corresponding system (2.12) & (2.7) on the coarse-grained level, and not vice versa, we have a working relation of cause and effect: During the coarse-graining process (i.e. in the limit of large $N \gg \tau$ and $m \gg l$) the variable transformation (2.7) on the fine-grained level (the cause) maintains to be a variable transformation also on the coarse-grained level (the effect), but which, as a pleasant side effect of this coarse-graining process, turns out to be additionally a *symmetry* transformation (which in this simple case, of course, has the exact same structure as the underlying transformation (2.7)). In other words, the statistical symmetry on the coarse-grained level (the effect) is linked to an underlying variable transformation on the fine-grained level (the cause), which itself, obviously, need *not* to be a symmetry.

[†]To avoid an unnecessary overload of formula referencing in the text and to therefore obtain a better readability, we will base the conclusion only on the probability distribution equations (2.2) and (2.12) of the random walk. It is obvious that this conclusion will then also hold identically for the associated and mathematically equivalent stochastic processes (2.1) and (2.11), respectively.

3. The conclusion in the general case

It's obvious that the above specific conclusion can be fully generalized: Formally, let F be any deterministic system, and let $\langle \cdot \rangle$ denote any coarse-graining or averaging process. Then we can make the following implications:

- I.1:** *Cause:* Let T be any variable transformation of F , i.e. $TF = \tilde{F}$
 \Rightarrow *Effect:* $T^* := \langle T \rangle$ is a variable transformation of $F^* := \langle F \rangle$, i.e. $T^*F^* = \tilde{F}^*$,

and if further the variable transformation commutes with the coarse-graining operator, i.e. $T^*\langle F \rangle = \langle TF \rangle$, then we have the additional entangled relation $\tilde{F}^* = \langle \tilde{F} \rangle = \tilde{F}^{*'}$. Amongst many possible transformations, this is in particular the case, if $T = S$ is a symmetry transformation, for which we then have the implication[†]

- I.2:** *Cause:* Let S be a symmetry transformation of F , i.e. $SF = F$
 \Rightarrow *Effect:* $S^* := \langle S \rangle$ is a symmetry transformation of $F^* := \langle F \rangle$, i.e. $S^*F^* = F^*$,

where, for example, the relation $S^*F^* = F^*$ can be used as a natural closure restriction, if the coarse-grained system F^* is unclosed. Now, it is obvious that the inverse or opposite conclusion of I.2 is false: A symmetry S^* on the coarse-grained level does not necessarily emerge from a symmetry S on the fine-grained level, as it was already shown in the above example of the diffusion equation and its underlying discrete random walk. But, since the inverse conclusion of I.1 is true, we have the combined implication

- I.3:** *Effect:* Let Σ^* be a symmetry transf. of $F^* := \langle F \rangle$, i.e. $\Sigma^*F^* = F^*$
 \Rightarrow *Cause:* At least one variable transf. Q of F must exist, such that $\langle Q \rangle = \Sigma^*$.

Hence, if such a variable transformation of *any* kind Q *cannot* be constructed, we have a violation of cause and effect, because we are dealing with a specific variable transformation Σ^* on the coarse-grained level which cannot be constructed from *any* kind of variable transformation on the fine-grained level. That means there is an effect without a cause, which is not physical. And exactly this is the case for the new statistical symmetries in Waławczyk *et al.* (2014), which constitutes the objection “*I. Violation of the causality principle*” in our Phys.-Rev.-E-Comment (Frewer *et al.*, 2015).

In general it is difficult to find or to construct a certain variable transformation Q on the fine-grained level such that its coarse graining turns the transformation into the given symmetry $\Sigma^* = \langle Q \rangle$. This is an inverse problem, for which in general no analytical solution in closed form may exist. But since all new statistical symmetries of Waławczyk *et al.* (2014), which were first determined and introduced in Oberlack & Rosteck (2010), are of a globally trivial form, it is analytically easy to show that no variable transformation of any kind on the fine-grained level in the sense of Q can be constructed at all. Every approach to construct such a transformation leads to a clear contradiction, as it was independently shown not only for the Lundgren-Monin-Novikov equations (see Frewer *et al.* (2015, 2014a)), but also for the Hopf equation (see Section 4.1 in Frewer *et al.* (2014b)), as well as for the multi-point correlation equations (see Sections 4.2 & 4.3 in Frewer *et al.* (2014b)). In particular for the new statistical scaling symmetry of the multi-point equations (Eq. (16) in Waławczyk *et al.* (2014)), it is straightforward to show that it's already by definition impossible to construct any kind of variable transformation for the instantaneous (fluctuating) velocity field such that it can

[†]Note that we only consider here the admitted symmetries of equations, and not of their solutions. Although a symmetry of an equation maps the set of *all* its solutions into itself, i.e., although one solution gets mapped into another solution, this symmetry, however, is not automatically admitted by a *particular* solution of this equation when emerging from specific initial or boundary conditions. The presence of such external conditions may not be compatible with the equations' symmetry, thus giving rise to solutions which do not reflect the symmetry of the equation itself. In the worst case, this symmetry is not even reflected in some asymptotic regime of the solution, which then is also known as the effect of spontaneous symmetry breaking.

induce this symmetry on the mean (averaged) level, where *all* velocity correlation functions $H_{\{n\}}$ are just multiplied by the *same* global factor e^{as} (see the proof e.g. in Appendix A of Frewer *et al.* (2014a)). The same issue we face for the coarse-grained multi-point PDFs (Eq. (63) in Waclawczyk *et al.* (2014)), where again *all* PDFs get scaled by this very same constant factor e^{as} , which, as before, simply cannot be linked to an underlying fine-grained variable transformation of *any* kind, and, therefore, inherently violates the classical principle of causality since there would be an effect without a cause. The same is also true for the new statistical translation symmetry, where *all* multi-point PDFs (Eq. (42) in Waclawczyk *et al.* (2014)) get translated by the *same* spatially global shift ψ .

Both these two new statistical symmetries are simply unphysical, as are also their consequences when trying to generate statistical scaling laws from them. As already said in the introduction, their unphysical behavior becomes clearly visible when comparing to DNS data (Simens *et al.*, 2009; Borrell *et al.*, 2013): Those scaling laws which involve any parameters from the unphysical symmetries, are completely unable to predict the correct DNS multi-point correlations behavior beyond the second order (Frewer *et al.*, 2014b). But, as soon as these unphysical group parameters are removed, the matching to the DNS data improves by several orders of magnitude and becomes well-defined again for all orders n , which definitely is a further and (from the analytical investigation) independent indication that the new statistical symmetries presented in Waclawczyk *et al.* (2014) are unphysical — here we refer to Frewer (2015) where we summarized our conclusion in more detail. Hence, as said before, these symmetries need to be discarded in our opinion, as they will only unnecessarily lead to erroneous conclusions in the theory of turbulence.

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